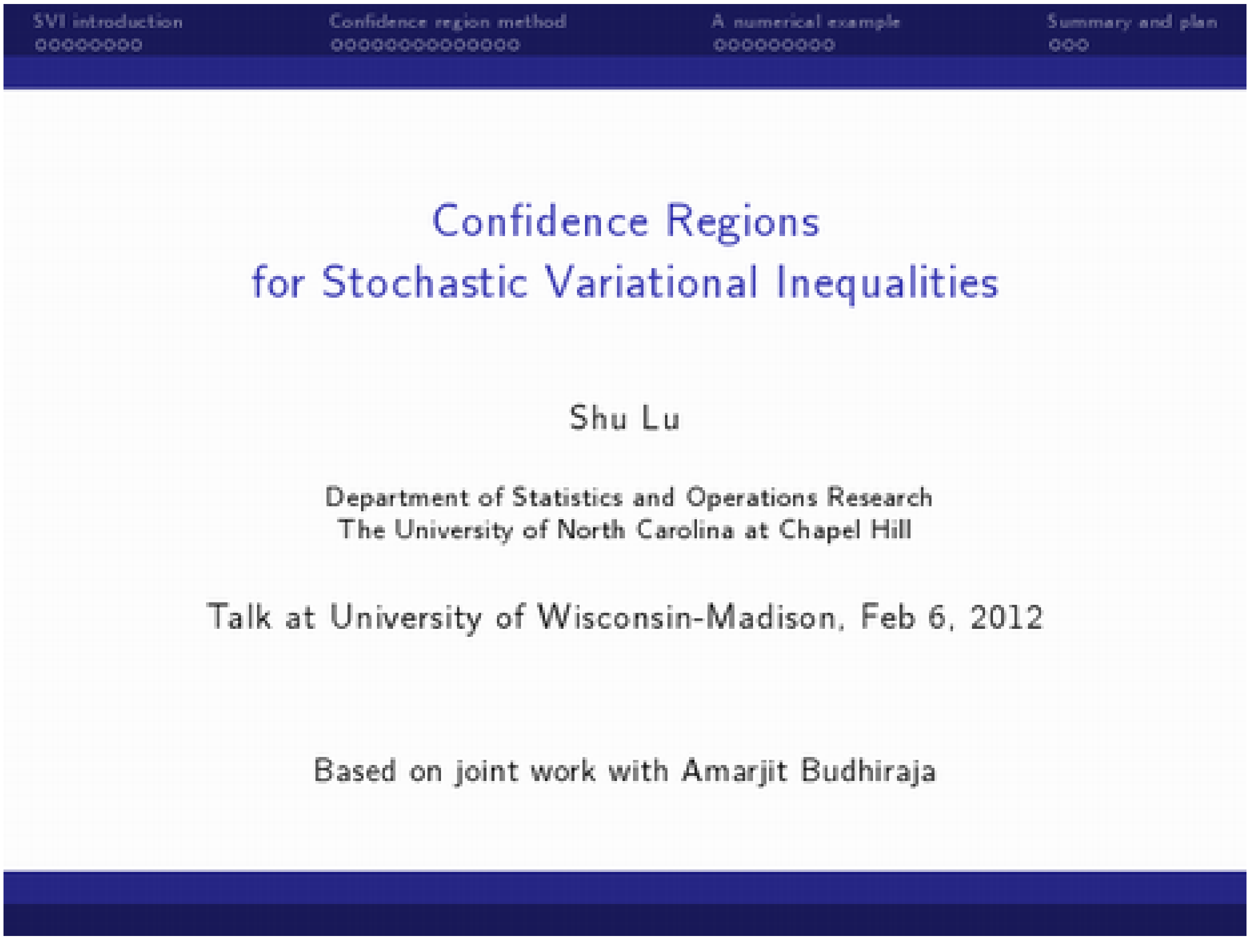

Confidence Regions for Stochastic Variational Inequalities

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Based on joint work with Amarjit Budhiraja

Outline

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An overview of related problems

Variational Inequalities

Stochastic Variational Inequalities

Stochastic Equilibria

Stochastic Optimization

Economic equilibria Nash equilibria userequilibria Frictional contactproblems Elastoplastic structure analysis Nonlinear obstacle problems

Data Uncertainty

Example: an SVI formulation of the lasso

$X \in \mathbb{R}^p$: a random input vector; $Y \in \mathbb{R}$: a random output variable

$\beta = (\beta_0, \beta_1, \cdots, \beta_p)$: the parameter to be determined

Let λ be a positive constant. A formulation of the lasso is

$$\min_{\beta} E\Big[Y - \beta_0 - \sum_{j=1}^{p} \beta_j X_j\Big]^2 + \lambda \sum_{j=1}^{p} |\beta_j|$$

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Define $S = \{(\beta, t) \in \mathbb{R}^{p+1} \times \mathbb{R}^p \mid t_j - \beta_j \geq 0, t_j + \beta_j \geq 0, j = 1, \cdots, p\}$

Under mild conditions, the lasso can be reformulated as

0 ∈ E

YX Y − − β

β

0

X 0

− − X X T(β T(β 1

1 ,⋯ ,⋯ ,β ,β p

p)

)X

+ N

S λe

(β,t)

where e ∈ Rp is all ones, and N

S

(β,t) is the normal cone to S at (β,t)

Example: an energy market problem

Players of the game: m producers/exporters of a type of energy

x_i: decision variable for player i (how much to invest, produce and ship ...), which takes values in a set K_i

θ_i: the profit of player i, a random function of $x = (x_1, \cdots, x_m)$

Each player selects x_i to maximize the expected profit $E(\theta_i)$

An equilibrium is attained, when no player can increase his expected profit by unilaterally changing his decision

Under mild conditions, the equilibrium problem can be formulated as

$$0 \in -E \begin{pmatrix} \nabla_{x_1} \theta_1(x) \\ \vdots \\ \nabla_{x_m} \theta_m(x) \end{pmatrix} + N_{K_1 \times \cdots \times K_m}(x)$$

(x)

Definition of stochastic variational inequalities

$(\Omega, \mathcal{F}, P)$: a probability space ξ: a random vector from Ω to a closed set $\Xi \subset \mathbb{R}^d$ O: an open subset of $\mathbb{R}^n$; X: a nonempty compact subset of O $F : O \times \Xi \to \mathbb{R}^n$, with $E\|F(x,\xi)\| < \infty$ for each $x \in O$ Let $f_0(x) = E[F(x,\xi)]$ for each $x \in O$ $S = \{x \in \mathbb{R}^n \mid Ax \le b\} = \{x \in \mathbb{R}^n \mid \langle a_i, x\rangle \le b_i, i = 1, \cdots, m\}$

The SVI problem is to find $x \in S \cap O$ such that

$$0 \in f_0(x) + N_S(x) \quad \text{(SVI)}$$

where $N_S(x)$ is the normal cone to S at x and is defined as

$$N_S(x) = \{v \in \mathbb{R}^n \mid \langle v, s - x\rangle \le 0 \text{ for each } s \in S\}$$

Example: a linear complementarity problem

Let $F : \mathbb{R}^2 \times \mathbb{R}^6 \to \mathbb{R}^2$ be defined by

$$F(x,\xi) = \begin{bmatrix} \xi_1 & \xi_2 \\ \xi_3 & \xi_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \xi_5 \\ \xi_6 \end{bmatrix},$$

where ξ is a random vector uniformed distributed on

$$\{\xi \in \mathbb{R}^6 \mid (0,0,0,0,-1,-1) \le \xi \le (2,1,2,4,1,1)\}$$

Let $S = \mathbb{R}^2_+$. The SVI problem becomes the following LCP:

$$0 \in \begin{bmatrix} 1 & 1/2 \\ 1 & 2 \end{bmatrix} x + N_{\mathbb{R}^2_+}$$

(x),

which has a unique solution x

0

= 0

The sample average approximation problem

In most problems of interest, f_0 does not have a closed form expression and requires a numerical approximation

Let $\xi^1,\cdots,\xi^N$ be i.i.d. r.v.s with distribution same as ξ

Define $f_N : O \times \Omega \to \mathbb{R}^n$ by $f_N(x,\omega) = N^{-1}\sum_{i=1}^{N} F(x,\xi^i(\omega))$

The SAA problem is to find $x \in S \cap O$ such that

$$0 \in f_N(x,\omega) + N_S(x) \quad \text{(SAA)}$$

For the LCP example, an SAA problem with $N = 10$ is given by

$$0 \in \begin{bmatrix} 0.9292 & 0.7536 \\ 0.5400 & 2.1111 \end{bmatrix} x + \begin{bmatrix} -0.1319 \\ -0.2906 \end{bmatrix} + N_{\mathbb{R}^2_+}(x),$$

which has a unique solution x_{10}

$= (0.0782, 0.1097)$